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LETTERS TO THE EDITOR

A CHECK ON THE ACCURACY OF TIMOSHENKO'S BEAM THEORY

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1. INTRODUCTION

Timoshenko's theory for vibrating beams extends the simple theory of beams to allow for their flexibility in shear and rotational inertia. Just as simple beam theory may cease to become accurate for very short beams, it may be anticipated that Timoshenko's theory loses its accuracy for vibrations with a very short wavelength.

An exact plane-stress solution is derived for the vibration of a very long rectangular strip of depth $2c$, giving a sinusoidal wave of wavelength $2\pi/\lambda$ at a frequency of $\omega/2\pi$ Hz. If the Poisson ratio is denoted by v, the shear modulus by G the density of the material by ρ, and γ is the non-dimensional parameter $\rho\omega^2/G\lambda^2$, simple beam theory gives the relationship

$$\gamma = \frac{2}{3}(1+v)\lambda^2 c^2 \tag{1}$$

and those given by Timoshenko's theory are

$$\gamma_1, \gamma_2 = (1+v) + \frac{5}{12} + \frac{5}{4\lambda^2 c^2} \pm \sqrt{\left[(1+v) + \frac{5}{12} + \frac{5}{4\lambda^2 c^2}\right]^2 - \frac{5}{3}(1+v)}, \tag{2}$$

where γ_1 will be taken as the lower of the two values (cf., reference [1] equation (61.74) for example).

2. THE PLANE-STRESS SOLUTION

The x-axis will be taken as lying along the middle line of the beam and the y-axis as normal to it. The material displacements u and v are in the x and y directions respectively. Assuming simple harmonic motion, the equations of dynamic equilibrium are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho\omega^2 = 0 \quad \text{and} \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho\omega^2 = 0, \tag{3}$$

where the stresses are given by

$$\sigma_{xx} = \frac{2G}{(1-v)}\left(\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right), \quad \sigma_{yy} = \frac{2G}{(1-v)}\left(\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial x}\right), \qquad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{4}$$

A solution will be sought of the form

$$u = \sum_i u_i \sinh m_i y \sin(\lambda x + \alpha)\sin(\omega t + \varepsilon), \qquad v = \sum_i v_i \cosh m_i y \cos(\lambda x + \alpha)\sin(\omega t + \varepsilon). \tag{5}$$

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Note that the symmetry about the x-axis ensures that if the free-surface conditions are satisfied on $y = c$, they will automatically be satisfied on $y = -c$.

Substituting equations (4) into equations (3) and using the expressions given by equations (5) yield the relationships between u_i and v_i given by

$$\begin{bmatrix} (1-\nu)(\mu_i^2+\gamma)-2 & -(1+\nu)\mu_i \\ (1+\nu)\mu_i & (1-\nu)(\gamma-1)+2\mu_i^2 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{6}$$

where $\lambda\mu_i$ has been substituted for m_i. For non-zero u_i and v_i, the determinant of the matrix in equation (6) must be zero, giving the two roots

$$\mu_1^2 = 1-\gamma \quad \text{and} \quad \mu_2^2 = 1-\frac{\gamma}{2}(1-\nu). \tag{7}$$

The two solutions which these roots generate can be used to satisfy the free-surface conditions that σ_{yy} and τ_{xy} are zero on $y = c$. Using the relationships between u_i and v_i given by equation (6), these conditions become

$$\begin{bmatrix} (2-\gamma)\cosh\mu_1\lambda c & 2\cosh\mu_2\lambda c \\ (2-2\gamma)\dfrac{1}{\mu_1\lambda c}\sinh\mu_1\lambda c & (2-\gamma)\dfrac{1}{\mu_2\lambda c}\sinh\mu_2\lambda c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8}$$

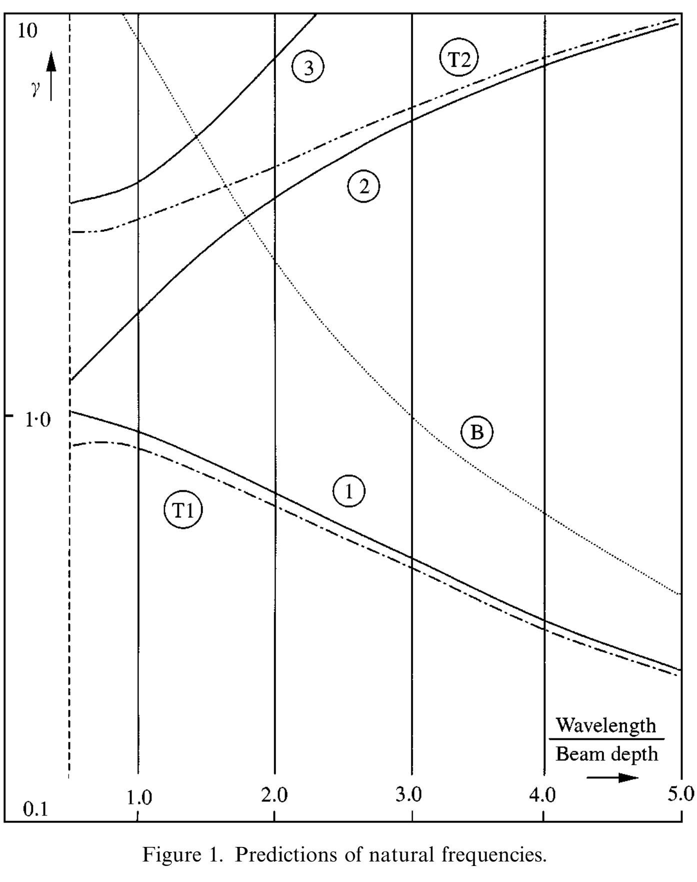

Figure 1. Predictions of natural frequencies.

As

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \cdots + \frac{x^{2n}}{2n!} \quad \text{and} \quad \frac{1}{x}\sinh x = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} \cdots + \frac{x^{2n}}{(2n+1)!} \tag{9}$$

the terms in equation (8) remain real, even for imaginary μ_i. The determinant of the matrix in equation (8) must be zero for non-zero v_1 and v_2. This yields a relationship between λc and γ.

Figure 1 shows the first three roots of γ for a range of wavelength/beam depth ($\pi/\lambda c$) values, taking v as 0·3. These curves are labelled with the numbers 1–3 in circles. Note that the vertical scale is logarithmic. The two roots of Timoshenko's solution given by equation (2) are shown by the broken curves labelled T1 and T2 and the simple beam solution given by equation (1) is given by the dotted line labelled B. It can be seen that the lower Timoshenko solution gives accurate values for wavelengths greater than the beam depth. When the two are equal, it underestimates the natural frequency by 5·8%. However, simple beam theory overestimates the natural frequency by 21·5% even when the wavelength is five times the beam depth.

An almost identical solution exists for travelling waves. In Figure 1, γ is then $\rho k^2/G$ where k is the wave velocity.

REFERENCE

1. W. FLÜGGE 1962 *Handbook of Engineering Mechanics*. New York: McGraw-Hill.